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Symbolic computing of nonlinear observable and observer forms ☆

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Abstract

Observer design for highly nonlinear dynamics is an important issue, particularly when the locally observable dynamics are not linearly observable. In such circumstances the ability to reduce the system to observable or observer form is a useful first step to observer design. We describe and illustrate symbolic computing tools to do that.

Keywords: Computer algebra; Observable form; Observer form; Nonlinear observer; Nonlinear systems

1. Introduction

When nonlinearities are essential, observability and observer design present new complexities that are absent in linear problems. Unlike linear systems, a

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nonlinear system may be observable for some inputs and not so for others. Reducing the system to observable or observer form can be a useful first step to nonlinear observer design, even for these more difficult situations. Computer tools to do this are the focus of this paper.

We begin with an overview of observability and a short and necessarily incomplete summary of nonlinear observer design methods is given in Section 2. An observability hierarchy is defined that progresses, in weakening degree, from 'linearly observable' to 'zero-input observable' to satisfaction of the 'observability rank condition' to 'locally observable.' The tools we describe herein apply to all of these cases. A new, general observable form for nonautonomous nonlinear systems is introduced and a multiple output generalization of the observer form construction given in [1] is given in Section 3. We also describe in detail the computations required to construct the observable and observer forms. In Section 3.5 we describe our implementation of them in *Mathematica*. Examples that illustrate all of the observability types in our hierarchy follow in Section 4. Our tools[1] extend the capabilities of the computations described in [2] to the multiple input/multiple output case.

2. Nonlinear system observability and observers

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i = f_u(x), \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in M$ (a neighborhood of x_0 in R^n), $u \in R^m$, and $y \in R^p$. We assume x_0 is an equilibrium point corresponding to zero input and output, i.e., $f(x_0) = 0$, $h(x_0) = 0$. The functions f, g_i, h are smooth. We write the right hand side of the differential equation as $f_u(x)$ to emphasize the role of u as a parameter of the vector field.

2.1. Observability

We very briefly summarize some well established facts about nonlinear system observability (see for example, [3,4]). The *observation space* $\mathcal{O}$ of system (1) is the linear space of functions $M \to R$ over the field R spanned by all functions of the form

$$L_{v_k} \cdots L_{v_1}(h_i), \quad k \geqslant 0, \ 1 \leqslant i \leqslant p, \ v_k, \ldots, v_1 \in \{f, g_1, \ldots, g_m\}. \tag{2}$$

[1] A copy of the *Mathematica* package can be obtained by contacting the first author.

It is important to emphasize that the observation space consists of all linear combinations of the functions (2) with real constant coefficients—viz., 'over the field R'. An analytic system (1) is observable on M if for any $x_1, x_2 \in M$, $x_1 \neq x_2$, there is a function $\Phi \in \mathcal{O}$ such that $\Phi(x_1) \neq \Phi(x_2)$. Associated with the observation space $\mathcal{O}$ is its differential $\mathrm{d}\mathcal{O}$, the codistribution

$$\mathrm{d}\mathcal{O} = \mathrm{span}\{\mathrm{d}\lambda | \lambda \in \mathcal{O}\}.$$

The *observability codistribution*, Ω_O, is the smallest codistribution that contains the covectors $\{\mathrm{d}h_1, \ldots, \mathrm{d}h_p\}$ and is invariant with respect to the vector fields $f, g_1, \ldots, g_m$. If $\mathrm{d}\mathcal{O}$ is nonsingular, then $\mathrm{d}\mathcal{O} = \Omega_O$.

The system is locally observable at x_0 if the observability codistribution, Ω_O has rank n at x_0. This is called the *observability rank condition*. If x_0 is a regular point of $\Omega_O(x_0)$, the observability rank condition is necessary as well as sufficient. If the system has zero input, then the observability codistribution reduces to

$$\Omega_L = \mathrm{span}\{L_f^k(\mathrm{d}h_i),\ 1 \leqslant i \leqslant p,\ 0 \leqslant k \leqslant n-1\}.$$

When $\dim \Omega_O(x_0) = n$ but $\dim \Omega_L(x_0) < n$, the implication is that some states are distinguishable only under the action of control inputs. When this occurs, most control inputs do distinguish the states. There are a few *singular inputs*, notably $u = 0$, that do not. Thus, when $\dim \Omega_L(x_0) = n$ we will use the terminology *observable for zero input at* x_0. It is also possible to test the linearization of (1) at x_0 for observability. That is, define

$$A_0 = \frac{\partial f}{\partial x}(x_0), \quad C_0 = \frac{\partial h}{\partial x}(x_0)$$

and test the pair (A_0, C_0). If the linearization is observable then we say that it is *linearly observable at* x_0. Linear observability implies zero-input observability. It is easy to prove that a system is linearly observable at x_0 if and only if $\dim \Omega_L(x_0) = n$. Thus, we have the following hierarchy

$$\begin{array}{ccc}
\dim \Omega_O(x_0) = n & \Rightarrow & \text{locally observable} \\
\Uparrow & & \Uparrow \\
\dim \Omega_L(x_0) = n & \Rightarrow & \text{zero} - \text{input observable} \\
\Updownarrow & & \Uparrow \\
\dim \begin{bmatrix} C_0 \\ C_0 A_0 \\ \vdots \\ C_0 A_0^{n-1} \end{bmatrix} = n & \Longleftrightarrow & \text{linearly observable.}
\end{array}$$

2.2. Approaches to nonlinear observer design

An observer for the system (1) is a dynamical system with inputs $y(\tau), u(\tau)$, $0 \leqslant \tau \leqslant t$ and output $\hat{x}(t) \in R^n$ such that $\hat{x}(t)$ is an estimate of $x(t)$ in the sense that $\|x(t) - \hat{x}(t)\| \to 0$ as $t \to \infty$. When (1) is linearly observable there are many approaches to observer design. On the other hand, if (1) is not linearly observable, options are limited. Observer design based on linearization up to output injection was introduced in [5,6] for the single-output case without inputs and extended to the multiple output case in [7]. In this approach the idea is to transform the system (1) into the so-called 'observer form'

$$\dot{z} = Az + \varphi(y), \quad y = Cz, \tag{3}$$

where A, C is an observable pair. When this is done, observer design is very easy. As might be expected, systems that can be transformed into the form (3) are rare but it is interesting to note that linear observability is not necessary if we do not insist that the transformation be a diffeomorphism. Xia and Zeitz [8] give conditions for observer construction for systems that are zero-input observable (see above hierarchy). This method (as do many others) begins with reduction of (1) to an 'observable form' that we will discuss below.

An extension to the case where (1) is not zero-input observable, is given by Hammouri and Gauthier [9,10]. The idea is to transform the system into the 'time varying' version of (3) specifically the 'observer form' given in (4), below.

$$\begin{aligned} \dot{z} &= A(u(t))z + \varphi(y, u(t)), \\ y &= [z_1 \quad \cdots \quad z_p]^{\mathrm{T}} = Cz. \end{aligned} \tag{4}$$

A constructive approach to computing the transformation for single-output systems is given in [1]. Recently, Souleiman et al. [11] present a different construction that applies to a somewhat larger class of single-output systems in which the matrix A is allowed to depend on both u and y. That is in (4) $A(u(t)) \to A(u(t), y(t))$.

For a system in the form of (4) a Kalman observer can be used

$$\begin{aligned} \dot{\hat{z}} &= A(u(t))\hat{z} + \varphi(u(t), y(t)) + P(t)C^{\mathrm{T}}(y(t) - C\hat{z}), \\ \dot{P} &= PA^{\mathrm{T}}(u(t)) + A(u(t))P - PC^{\mathrm{T}}CP + Q. \end{aligned}$$

This observer converges exponentially provided $u(t)$ is such that the linear time-varying system

$$\dot{z} = A(u(t))z, \quad y = Cz$$

is completely observable [12]. This 'passive approach' relies on the natural occurrence of a suitably rich input.

3. Observable and observer forms

When (1) is not linearly observable, but nonetheless locally observable, we need to be able to reduce the system to either observable or observer form as a first step to observer design using existing methods. Even for linearly observable systems this may be a convenience. In this section we describe the computations needed to do that.

3.1. Control sequences

One characterization of the observation space is given by the following result.

Lemma 3.1. *The observation space $\mathcal{O}$ is equivalent to the linear vector space of functions $M \to R$ over the field R*

$$\widetilde{\mathcal{O}} = \mathrm{span}_R\left\{ L_{f_{u^k}} \cdots L_{f_{u^1}}(h_i) \middle| \begin{array}{l} 1 \leqslant i \leqslant p,\ k \geqslant 0, \\ u^1, \ldots, u^k \in \{0,1\}^m \end{array} \right\}.$$

Proof. The result follows from direct calculations based on the 'linear in control' structure of (1). □

This motivates the following definitions. Define a sequence of codistributions

$$\mathscr{E}_0 := \mathrm{span}\{\mathrm{d}h\},$$
$$\mathscr{E}_k = \mathscr{E}_{k-1} + \mathrm{span}\{\mathrm{d}L_{f_{u^k}} \cdots L_{f_{u^1}}(h) | u^i \in \{0,1\}^m,\ i = 1, \ldots, k\}.$$

We assume that, 'almost everywhere' on a neighborhood of x_0, (i) the codistributions $\mathscr{E}_k$ are of constant dimension, and (ii) there exists a smallest p^* such that

$$\mathscr{E}_0 \subset \cdots \subset \mathscr{E}_{p^*} = \mathscr{E}_{p^*+1} = \mathrm{d}\mathcal{O}.$$

Let n_k denote the codimension of $\mathscr{E}_{k-1}$ in $\mathscr{E}_k$. Then there exists sets of control sequences [1]

$$I_1 = \{(u^{i_1}) | u^{i_1} \in \{0,1\}^m\},$$
$$I_2 = \{(u^{i_1}, u^{i_2}) | u^{i_1} \in \{0,1\}^m,\ u^{i_2} \in \{0,1\}^m\},$$
$$\vdots$$

that satisfy

(a) If $(u^{i_1}, \ldots, u^{i_j}) \in I_j$ then $(u^{i_1}, \ldots, u^{i_{j-1}}) \in I_{j-1}$, for $j \geqslant 2$.

(b) The one-forms

$$\bigcup_{l=1}^{k}\{\mathrm{d}L_{f_{u^l}}\cdots L_{f_{u^1}}(h)|(u^{i_1},\ldots,u^{i_l})\in I_l\}\cup\{\mathrm{d}h\}$$

span $\mathscr{E}_k$ on a neighborhood of x_0. In the single-output case these one-forms actually constitute a basis for $\mathscr{E}_k$ and the cardinal number of I_k is n_k.

We obtain the control sequences, I_k, by direct, sequential construction of the codistributions $\mathscr{E}_k$. See [1] for more details about the single-output case.

Example 3.2. Consider the 5th order system

$$\dot{x}=\begin{bmatrix} \mathrm{e}^{x_1+x_2}-1+ux_1^2 \\ -\mathrm{e}^{x_1+x_2}+1+u(\mathrm{e}^{x_3-x_2}-\mathrm{e}^{-x_1-x_2}-x_1^2) \\ -\mathrm{e}^{x_1+x_2}+1+x_1^3\mathrm{e}^{-x_1-x_3}-ux_1^2 \\ x_5 \\ x_1 \end{bmatrix}, \quad y=\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}.$$

This system is locally observable, but it is not observable for all inputs. In particular, it is not observable with $u\equiv 0$. We use the *Mathematica* function ControlSequences (see Section 3.5) to compute

$$\mathscr{E}_0=\{\mathrm{d}[x_1],\mathrm{d}[x_4]\},$$
$$\mathscr{E}_1=\{\mathrm{d}[x_1],\mathrm{d}[x_2],\mathrm{d}[x_4],\mathrm{d}[x_5]\},$$
$$\mathscr{E}_2=\{\mathrm{d}[x_1],\mathrm{d}[x_2],\mathrm{d}[x_3],\mathrm{d}[x_4],\mathrm{d}[x_5]\},$$

and

$$I_1=\{0\},\quad I_2=\{0,1\}.$$

Now, direct computation leads to the four one-forms

$$\alpha=\mathrm{d}[L_{f_{u=0}}(h_1)]=\mathrm{e}^{x_1+x_2}(\mathrm{d}[x_1]+\mathrm{d}[x_2]),$$
$$\beta=\mathrm{d}[L_{f_{u=0}}(h_2)]=\mathrm{d}[x_5],$$
$$\gamma=\mathrm{d}[L_{f_{u=1}}L_{f_{u=0}}(h_1)]=\mathrm{e}^{x_1+x_3}(\mathrm{d}[x_1]+\mathrm{d}[x_3]),$$
$$\delta=\mathrm{d}[L_{f_{u=1}}L_{f_{u=0}}(h_2)]=\mathrm{d}[x_1],$$

and

$$\mathrm{span}\{\mathrm{d}[x_1],\mathrm{d}[x_4],\alpha,\beta\}=\mathrm{span}\{\mathrm{d}[x_1],\mathrm{d}[x_2],\mathrm{d}[x_4],\mathrm{d}[x_5]\}=\mathscr{E}_1,$$
$$\mathrm{span}\{\mathrm{d}[x_1],\mathrm{d}[x_4],\alpha,\beta,\gamma,\delta\}=\mathrm{span}\{\mathrm{d}[x_1],\mathrm{d}[x_2],\mathrm{d}[x_3],\mathrm{d}[x_4],\mathrm{d}[x_5]\}=\mathscr{E}_2.$$

3.2. Observability indices

Consider the set of I_j consisting of n_j j-tuples:

$$I_j=\{(u^{i_1,1},\ldots,u^{i_j,1}),\ldots,(u^{i_1,n_j},\ldots,u^{i_j,n_j})\}$$

and define the $p \cdot n_j$-vector of jth order Lie derivatives

$$L_{f_{I_j}}(h) = \begin{bmatrix} L_{f_u^{i_{1,1}}} \cdots L_{f_u^{i_{j,1}}}(h) \\ \vdots \\ L_{f_u^{i_{1,n_j}}} \cdots L_{f_u^{i_{j,n_j}}}(h) \end{bmatrix}.$$

Now, consider the collection of covectors $\mathrm{d}L_{f_{I_i}}(h)$ for $i = 0,\ldots,p^*$, which we can arrange in the (block) tableau

$$\begin{matrix} \mathrm{d}h_1 & \mathrm{d}h_2 & \cdots & \mathrm{d}h_p \\ \mathrm{d}L_{f_{I_1}}(h_1) & \mathrm{d}L_{f_{I_1}}(h_2) & \cdots & \mathrm{d}L_{f_{I_1}}(h_p) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{d}L_{f_{I_{p^*}}}(h_1) & \mathrm{d}L_{f_{I_{p^*}(h_2)}} & \cdots & \mathrm{d}L_{f_{I_{p^*}}}(h_p) \end{matrix}.$$

From this set we seek to identify a maximal set of independent covectors. We can do this by searching down columns or across rows (recall the linear counterpart). For a row search, begin with the first row and work from left to right, then move to the next row. If the outputs are them selves independent, we identify p chains of covectors $\mathrm{d}h_i \mathrm{d}L_{f_i^1}(h_i) \cdots \mathrm{d}L_{f_i^{\kappa_1 - 1}}(h_i)$ of length κ_i, $i = 1,\ldots,p$. The integers κ_i are the *observability indices*. For an observable system $\kappa_1 + \kappa_2 + \cdots + \kappa_p = n$. For autonomous systems this definition of observability indices is equivalent to that in [13,14].

3.3. Observable form

If the system is observable, then we can define new state variables $z \in R^n$ via the transformation $x \to z$.

$$z = \begin{bmatrix} h_1 \\ \vdots \\ L_{f_1^{k_1 - 1}}(h_1) \\ \vdots \\ h_p \\ \vdots \\ L_{f_p^{k_p - 1}}(h_p) \end{bmatrix}. \tag{5}$$

If the inverse is continuous and the transformed equations produce unique solutions we call the transformed equations an *observable form*. This is consistent with the usual terminology for linear systems and autonomous nonlinear systems. In the latter case, the transformed equations are in the form of p chains,

$$\begin{array}{ccc} \dot{z}_1 = z_2 & \cdots & \dot{z}_{\kappa_1+\cdots+\kappa_{p-1}+1} = z_{\kappa_1+\cdots+\kappa_{p-1}+2} \\ \vdots & \cdots & \vdots \\ \dot{z}_{\kappa_1-1} = z_{\kappa_1} & \cdots & \dot{z}_{\kappa_1+\cdots+\kappa_p-1} = z_{\kappa_1+\cdots+\kappa_p} \\ \dot{z}_{\kappa_1} = \varphi_1(z) & \cdots & \dot{z}_{\kappa_1+\cdots+\kappa_p} = \varphi_p(z) \\ y_1 = z_1 & \cdots & y_p = z_{\kappa_1+\cdots+\kappa_{p-1}+1} \end{array}.$$

Remark 3.3 (*Xia and Zeitz*). Note that if

$$\operatorname{rank}\begin{bmatrix} \mathrm{d}h_1 \\ \vdots \\ \mathrm{d}L_{f_1^{k_1-1}}(h_1) \\ \vdots \\ \mathrm{d}h_p \\ \vdots \\ \mathrm{d}L_{f_p^{k_p-1}}(h_p) \end{bmatrix}(x_0) = n$$

the implicit function theorem guarantees the existence of a smooth (local) inverse of the transformation (5) so that the transformation is a diffeomorphism. However, an inverse may exist even if the rank condition fails. In this case, the inverse will only be continuous. If the transformed differential equations have unique solutions on a neighborhood of x_0, then this is still a useful transformation. This point is described more fully in Xia and Zeitz [8].

Example 3.4 (*Continuation of Example* 3.2). Let us return to Example 3.2 and compute the observable form. We find the observability indices to be 3, 2. The transformation to observable form is

$$z_1 = x_1, \quad z_2 = -1 + \mathrm{e}^{x_1+x_2}, \quad z_3 = -1 + \mathrm{e}^{x_1+x_3}, \quad z_4 = x_4, \quad z_5 = x_5$$

from which the observable form is obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} z_2 + uz_1^2 \\ uz_3 \\ z_1^3 \\ z_5 \\ z_1 \end{bmatrix}, \quad y = \begin{bmatrix} z_1 \\ z_4 \end{bmatrix}.$$

In this example, we see the two-chain structure of the observable form equations. Also, the role of the control input is displayed. We see clearly that the system is not observable if $u \equiv 0$.

3.4. Observer form

Now, we consider transforming (1) into the special form (time-varying linear up to output injection) (4). In this form it is possible to use linear methods for observer design. Eq. (4) will be called an *observer form* of which (3) is a special case. Not every locally observable nonlinear system (1) has an observer form.

The formulation we follow is that of [9,1,10]. First, let us introduce some definitions. Consider a set of p vector fields, $X = \{X_1, \ldots, X_p\}$. Sequentially define sets of $p + 1$-forms

$$\begin{aligned} \Omega_1^X &= \mathrm{span}_R\{\mathrm{d}L_{f_u}(h_i)\wedge_{j=1}^{p}\mathrm{d}h_j | i = 1, \ldots, p; u \in \{0,1\}^m\}, \\ \Omega_{k+1}^X &= \mathrm{span}_R\{\mathrm{d}L_{f_u}(i_X\alpha)\wedge_{j=1}^{p}\mathrm{d}h_j | \alpha \in \Omega_k^X, u \in \{0,1\}^m\}, \\ \Omega^X &= \sum_{k\geqslant 1}\Omega_k^X. \end{aligned}$$

Let $i_f(\omega)$ denote the usual contraction of the form ω with respect to the vector field f. We use the notation

$$i_X(\omega) = i_{X_1} \circ \cdots \circ i_{X_p}(\omega).$$

The following proposition, given in [10], generalizes the single-output result in [9] to multiple outputs.

Proposition 3.5. *The system* (1) *is transformable into the observer form* (4) *if and only if*:

(1) $\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_p(x_0) \neq 0$ (*independent outputs*).
(2) *There exists a set of vector fields* $X_1, \ldots, X_p$ *that satisfies*
  (a) $L_{X_i}h_j = \delta_{ij}$
  (b) $\dim \Omega^X = n - p$
  (c) $\forall\omega \in \Omega^X, \mathrm{d}i_X(\omega) = 0$
  (d) $i_X(\omega_1) \wedge \cdots \wedge i_X(\omega_{n-p}) \wedge \mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_p(x_0) \neq 0$ *where* $\omega_j\, j = 1, \ldots, n - p$ *is any basis for* Ω^X.

If these conditions hold, then the transformation is given by

$$\begin{aligned} z_1 &= h_1(x), \ldots, z_p = h_p(x), \\ \mathrm{d}z_{j+p} &= i_X(\omega_j), \quad j = 1, \ldots, n - p. \end{aligned} \tag{6}$$

Proof. A sketch of the proof is given in [10]. However, it is useful here to provide a more complete discussion of the sufficiency part in order to clarify the nature of later computations. It is provided in Appendix A. □

Let us make a few comments about the stated conditions.

Remark 3.6 (*Concerning item* 2).

(1) Item (a) implies the p-tuple of vector fields $X = [X_1, \ldots, X_p]$ forms a right inverse of the Jacobian $\partial h/\partial x$. The vector field X_i is aligned with the direction of $y_i = h_i$, that is, it is orthogonal to the codimension-1 surfaces $h_i(x) = \text{constant}$. As in the single-output case [15] every X_i satisfying the conditions of Proposition 3.5 is a constant vector field in the linearized coordinates, so it takes the form:

$$X_i = \frac{\partial}{\partial z_i} + c^i_{p+1}\frac{\partial}{\partial z_{p+1}} + \cdots + c^i_n \frac{\partial}{\partial z_n}, \quad i = 1, \ldots, p.$$

Furthermore, a linear change of coordinates $\tilde{z} = Tz$ with $\tilde{z}_i = z_i$, $i = 1, \ldots, p$ leaves the equations in TVLOI form and one can be found so that

$$X_i = \frac{\partial}{\partial \tilde{z}_i}.$$

Such X_i satisfy the conditions of Proposition 3.5 for a system in the form observer form (4), [9,10].

(2) In item (b) Ω^X is considered a vector space over the reals. So $\dim \Omega^X = \dim \operatorname{span}_R \Omega^X$, which is not the same as $\dim \operatorname{span} \Omega^X$. See item (5) below.

(3) Item (c) is an integrability condition for each of the 1-forms $i_X(\omega)$. Recall that any 1-form has the representation

$$\omega = \sum a_i(x)\mathrm{d}x_i.$$

Thus, we have the differential

$$\mathrm{d}\omega = \sum_i \mathrm{d}a_i \wedge \mathrm{d}x_i = \sum_{i,j} \frac{\partial a_i}{\partial x_j}\mathrm{d}x_j \wedge \mathrm{d}x_i = 0.$$

Since $\mathrm{d}x_j \wedge \mathrm{d}x_i = -\,\mathrm{d}x_i \wedge \mathrm{d}x_j$, this implies that the Jacobian $\partial a/\partial x$ is symmetric so that the 1-form $d\omega$ is an exact differential.

(4) Item (d) implies that the n coordinate functions

$$z_1(x) = h_1(x), \ldots, z_p(x) = h_p(x), z_p(x), \ldots, z_n(x)$$

are independent thereby defining a valid coordinate transformation.

(5) Items (b) and (d) together imply $\dim \operatorname{span} \Omega^X = n - p$. This follows from the fact item (d) requires $i_X(\omega_1)\wedge \cdots \wedge i_X(\omega_{n-p}) \neq 0$ for every basis $\{\omega_1,\ldots,\omega_{n-p}\}$ of Ω^X, i.e., the 1-forms $i_X(\omega_1),\ldots,i_X(\omega_{n-p})$ are independent in the usual sense (field of admissible functions). But this can be true only if the p-forms ω_i are independent. To see this, suppose X is a vector field on R^n and suppose ω_1, ω_2 are p-forms. Define a third p-form that is dependent on ω_1 and ω_2, $\omega_3 = \gamma_1(x)\omega_1 + \gamma_2(x)\omega_2$. Then $i_X(\omega_3) = \gamma_1(x)i_X(\omega_1) + \gamma_2(x)i_X(\omega_2)$. Consequently,

$$i_X(\omega_1) \wedge i_X(\omega_2) \wedge i_X(\omega_3) \\ = \gamma_1(x)i_X(\omega_1) \wedge i_X(\omega_2) \wedge i_X(\omega_1) + \gamma_2(x)i_X(\omega_1) \wedge i_X(\omega_2) \wedge i_X(\omega_2) = 0.$$

This calculation extends to the general case in which there is a dependence among any number of p-forms.

Now, we need to provide a construction for the set of vector fields X. First, obtain a set of vector fields $Y_1,\ldots, Y_p$ that satisfy

$$\begin{bmatrix} \mathrm{d}h_1 \\ \vdots \\ \mathrm{d}L_{f_1}^{\kappa_1-1}(h_1) \\ \vdots \\ \mathrm{d}h_p \\ \vdots \\ \mathrm{d}L_{f_p}^{\kappa_p-1}(h_p) \end{bmatrix} [Y_1 \quad \cdots \quad Y_p] = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & & 0 \\ 0 & & \vdots \\ 0 & \cdots & \vdots \\ \vdots & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}. \tag{7}$$

For any control sequence $u^1, u^2, \ldots$ we can define the set of vector fields

$$Z^i_{u^1\cdots u^{\kappa_i-1}} = [f_{u^{\kappa_i-1}}, [\cdots [f_{u^1}, Y_i]\cdots]], \quad i = 1,\ldots,p.$$

Now, identify the subset of control sequences $\mathscr{I} \subseteq I_{p^*}$ that satisfy

$$\det \left[L_{Z^1_{u^1\cdots u^{\kappa_1-1}}}(h) \cdots L_{Z^p_{u^1\cdots u^{\kappa_p-1}}}(h) \right] \neq 0$$

and use any one of these sequences to obtain

$$[X_1 \quad \cdots \quad X_p] = \left[L_{Z^1_{u^1\cdots u^{\kappa_1-1}}}(h) \quad \cdots \quad L_{Z^p_{u^1\cdots u^{\kappa_p-1}}}(h) \right]^{-1} \\ \times \left[Z^1_{u^1\cdots u^{\kappa_1-1}} \quad \cdots \quad Z^p_{u^1\cdots u^{\kappa_p-1}} \right]. \tag{8}$$

The following theorem summarizes the key result. It generalizes the single-output case proved in [1].

Proposition 3.7. *The system* (1) *is transformable into the observer form* (4) *if and only if*:

(1) $\mathscr{I} \neq \emptyset$.
(2) $\forall (u_1, \ldots, u_{p*}) \in \mathscr{I}$, $\mathrm{d}L_{Z_{u^1 \cdots u^{\kappa_i - 1}}}(h_i) = 0$, $i = 1, \ldots, p$.
(3) *The set of vector fields* $X_1, \ldots, X_p$ *is given by Eq.* (8), *and the following conditions hold*:
   (a) $\dim \Omega^X = n - p$.
   (b) $\forall \omega \in \Omega^X, \mathrm{d}i_X(\omega) = 0$.
   (c) $i_X(\omega_1) \wedge \cdots \wedge i_X(\omega_{n-p}) \wedge \mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_p(x_0) \neq 0$ *where* ω_j $j = 1, \ldots, n - p$ *is any basis for* Ω^X.

Proof. *Sufficiency* follows from Proposition 3.5. *Necessity* is proved in Appendix B. □

3.5. Computational tools

The computations described above have been implemented in a *Mathematica* package. In this section, we wish to summarize the key elements of our implementation.

The package has three primary high level functions:

(1) ObservabilityIndices, computes the observability indices.
(2) ObservableTransform, computes the transformation to observable form.
(3) LinearizeToOutputInjection, computes the transformation to observer form.

These are supported by several utility functions that compute the control sequences, solve the first order partial differential equations of Proposition 3.5, and others. The most important of these are

(1) ControlSequences
(2) OmegaForms
(3) SpanR

Underlying these calculations are basic tools for working with differential forms. We have slightly extended the Exterior Differential Calculus package of [16]. These three new tools have been incorporated into the *ProPac* package described in [17].

A key construction is ControlSequences which performs the computations outlined in Section 3.1. The algorithm proceeds as follows.

Algorithm (*ControlSequences*).

Input: f, h, x, u ($\dot{x} = f_u(x), y = h(x)$)
Output: list of indices, n_k, list of sets of control sequences, I_k
begin
$\mathscr{E}_0 = \{\mathrm{d}h\}$; $r = \dim\mathscr{E}_0$; $k = 0$;
while ($\dim\mathscr{E} < n$)&&($k < n$) do
  k++
  Set up $\mathscr{E}_k = \left\{\mathrm{d}[L_{f_{u^k}} \cdots L_{f_{u^1}}(h)] \cup \mathrm{d}h\right\}$ with generic control sequence
  $n_k := \dim\mathscr{E}_k - \dim\mathscr{E}_{k-1}$;
  Enumerate all controls $u^k \in \{0,1\}^m$ that do not reduce $\dim\mathscr{E}_k =: \mathscr{U}_k$
  Pick out n_k control sequences of the form
  $s_k = \{s_{k-1}, u^k\},\ u^k \in \mathscr{U}_k, s_{k-1} \in I_{k-1} =: I_k$
end

Once the control sequences are obtained, it is a simple mater to set up and solve Eqs. (7) and (8). Once the vector fields $X_1,\ldots,X_p$ are obtained, we compute Ω^X using the function OmegaForms.

Algorithm (*OmegaForms*).

Input: f, h, x, u, $X_1,\ldots,X_p$
Output: a basis for Ω^X
begin
$\Omega_1^X = \mathrm{span}_R\left\{\mathrm{d}L_{f_u}(h_i)\wedge_{j=1}^{p}\mathrm{d}h_j \,\middle|\, \begin{matrix} i = 1,\ldots,p \\ u \in \{0,1\}^m \end{matrix}\right\}$;
$\Omega^X = \Omega_1^X$;
$k = 1$;
while $\dim\Omega^X < n-p$ do
  k++
  $\Omega_k^X = \mathrm{span}_R\left\{\mathrm{d}L_{f_u}(i_X(\alpha))\wedge_{j=1}^{p}\mathrm{d}h_j \,\middle|\, \begin{matrix} \alpha \in \Omega_{k-1}^X \\ u \in \{0,1\}^m \end{matrix}\right\}$
  $\Omega^X := \Omega^X + \Omega_k^X$
end

The central calculations in the above procedure are the summation in the last step and the construction span_R. The summation is based on item (4) of Remark 3.6. We successively check each $(p+1)$-form $\alpha \in \Omega_k^X$. If $\alpha \in \mathrm{span}\Omega^X$ we drop it, otherwise we join it to the set of $(p+1)$-forms that define Ω^X.

Now, consider the procedure for computing span_R.

Algorithm (*SpanR*).

> **Input**: a list of n forms of dimension p, $A = \{\alpha_1, \ldots, \alpha_n\}$
> **Output**: a set of basis forms for $\text{span}_R A$
> begin
> Basis = $\{\alpha_1\}$ (assuming α_1 is not trivial)
> $k = 2$
> while $k \leqslant n$ do
> k++
>   Check if α_k can be expressed as a linear combination, over the reals, of the forms in Basis. If not add α_k to Basis.
> end

The test in the above algorithm is implemented using the *Mathematica* function Reduce. Suppose, at the kth step, we have

$$\text{Basis} = \{\beta_1, \ldots, \beta_q\}.$$

We want to determine if there exists real numbers $k_1, \ldots, k_q$ such that

$$\alpha_k = k_1\beta_1 + \cdots + k_q\beta_q.$$

Reduce allows us to seek solutions of this equation with the unknowns $k_1, \ldots, k_q$ restricted to real numbers.

4. Examples

Several examples follow that illustrate the computations (these and other examples can be found worked out in the *Mathematica* notebook Examples.nb that can be downloaded from http://www.pages.drexel.edu/~hgk22/notebook.htm). In each case we compute both the observable and observer forms. First, Example 4.8 is linearly observable (and therefore zero-input observable). Example 4.9 is zero-input observable but not linearly observable. Example 4.10 is locally observable but does not satisfy the observability rank condition. Example 4.11 is linearly observable. Example 4.12 is not zero-input observable but satisfies the observability rank condition.

Example 4.8 (*Krener and Respondek Example* 7.3). Consider the three state, nonautonomous system from [7]

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 + (1 + \mathrm{e}^{x_1})u \\ 3x_1^2x_2^2 + x_1^3x_3 + (1 + x_1 + x_2)u \end{bmatrix}, \quad y = x_1.$$

This system is linearly observable. Notice that it is already in observable form. Applying the tools described above, we find that the system transforms to observer form with the transformation

$$z_1 = x_1, \quad z_2 = (x_1^4 - 4x_2)/2, \quad z_3 = -x_1^3 x_2 + x_3.$$

The observer form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -2 \\ 0 & -\frac{1}{2}u & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{4}y^4 \\ -2(1+\mathrm{e}^y)u \\ (1+y-(1+\mathrm{e}^y)y^3 + y^4/4)u \end{bmatrix}.$$

Example 4.9 (*Xia and Zeitz Example* 2). Now, consider the simple two state, single output, autonomous example from [8]. Although the transformation is smooth, its inverse is only continuous.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = x_1^2 + x_2^5.$$

The system is observable with index 2, but it is not linearly observable. The transformation to observable form is smooth

$$z_1 = x_1^3 + x_2^5, \quad z_2 = 3x_1^3 + 5x_2^5.$$

But its inverse is not

$$x_1 = -\left(-\frac{1}{2}\right)^{1/3}(5z_1 - z_2)^{1/3},$$

$$x_2 = -\left(-\frac{1}{2}\right)^{1/5}(-3z_1 + z_2)^{1/5}.$$

The observable form equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -15z_1 + 8z_2 \end{bmatrix}, \quad y = z_1.$$

The transformation to observer form is

$$z_1 = x_1^3 + x_2^5, \quad z_2 = 5x_1^3 + 3x_2^5$$

and its inverse is

$$x_1 = -\left(-\frac{1}{2}\right)^{1/3}(-3z_1 + z_2)^{1/3},$$

$$x_2 = -\left(-\frac{1}{2}\right)^{1/5}(5z_1 - z_2)^{1/5}.$$

The observer form equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 8z_1 - z_2 \\ 15z_1 \end{bmatrix} = \begin{bmatrix} -z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 15 \end{bmatrix} y.$$

Example 4.10 (*Xia and Zeitz Example* 3). Now consider a nonautonomous example, from [8]. It is not zero-input observable. However, it is observable with observability index 3. As we will see, the observable and observer form are the same. The transformation is smooth, but its inverse is merely continuous.

$$\dot{x}_1 = x_2^3,$$
$$\dot{x}_2 = x_2 u,$$
$$y = x_1.$$

The transformation to observable/observer form is

$$z_1 = x_1, \quad z_2 = -x_2^3$$

and its inverse is

$$x_1 = z_1, \quad x_2 = -z_2^{1/3}.$$

The transformed equations are

$$\dot{z}_1 = -z_2,$$
$$\dot{z}_2 = 3z_2 u,$$
$$y = z_1.$$

Example 4.11 (*Hou and Pugh*). This example is from [18]. They propose a method for linearization to output injection for multiple output autonomous systems different from that implemented here. To obtain the observer form we need to reorder the outputs.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 x_3 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The system is observable with indices 2, 1. The transformation to observable form is simply a reordering of states

$$z_1 = x_3, \quad z_2 = x_2, \quad z_3 = x_1$$

leading to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_1 z_2 \\ z_2 \end{bmatrix}.$$

The transformation to observer form is

$$z_1 = x_3, \quad z_2 = x_1, \quad z_3 = \tfrac{1}{2}(-2x_2 + x_3^2).$$

This transformation produces the observer form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(z_1^2 - 2z_3) \\ \frac{1}{2}(z_1^2 - 2z_3) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \frac{z_1^2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$y = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Example 4.12 (*Continuation of Example* 3.2). We return to Example 3.2 and compute the observer form. The transformation to observer form is found to be

$$z_1 = x_1, \quad z_2 = x_4, \quad z_3 = -\mathrm{e}^{x_1+x_2}, \quad z_4 = -x_5, \quad z_5 = \mathrm{e}^{x_1+x_3}.$$

From this we find the observer form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} -z_3 - 1 + uz_1^2 \\ -z_4 \\ u(1 - z_5) \\ -z_1 \\ z_1^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} + \begin{bmatrix} -1 + y_1^2 u \\ y_2 \\ u \\ -y_1 \\ y_1^3 \end{bmatrix}.$$

5. Conclusions

The observability properties of nonlinear systems have nuances that have no counterpart in linear theory. One consequence of this is that there are opportunities for state estimation in nonlinear systems even when its linearization is not observable or there is some other pathology associated with observability (see Section 2). There are important practical implications because problems like this occur when operating around bifurcation points and in fault detection and identification. However, to take advantage of these possibilities it is necessary to build observers using new design paradigms, some of which have emerged in recent years. To do so requires development of new computational tools.

In this paper, we have described symbolic computations for reducing nonlinear smooth affine systems to observable and observer forms, when possible, as the first step in observer design. These tools can be applied to systems that are linearly observable, locally observable with zero input or merely locally observable. Our approach involves computations with differential forms, which experience shows to be extremely efficient.

Our characterization has at its root the computation of sequences of constant controls as formulated in [1]. This idea appears to have its origins in the pioneering work of [19]. Using this construction, we introduce a local observable form for nonautonomous systems that is consistent with prior work and complements the observer form of [10]. Our approach to computing the observer form is based on a multiple-output generalization (Proposition 3.7) of the method proposed in [1].

In Section 3.5 we describe the essential computations in our implementation and in Section 4 we give several examples. The examples are chosen to illustrate a variety of circumstances. The following cases are covered:

(1) autonomous and nonautonomous,
(2) linearly observable,
(3) not linearly observable, but zero-input observable,
(4) not zero-input observable, but satisfies the observability rank condition,
(5) locally observable, but does not satisfy the observability rank condition.

There are further enhancements that need to be considered. Of course, not all locally observable systems have an observer form. However, the class of systems that do can be expanded if one allows for a transformation of the outputs. This was pointed out in [5]. In the single-output case, necessary conditions for the output transformation where obtained by Besancon [20] in the framework employed herein. In the multiple-output case even output reordering helps (see Example 4.11 and [18]).

Appendix A. Proof of sufficiency part of Proposition 3.5

Sufficiency: Assume that the hypotheses of the proposition hold, and the new coordinates are defined by Eqs. (6). The vector field f_u can be expressed in the new coordinates

$$f_u = \sum_{i=1}^{n} L_{f_u}(z_i)\frac{\partial}{\partial z_i}.$$

We wish to determine the components of the vector field $F_u^i = L_{f_u}(z_i)$. Notice that, for $i > p$, we can write

$$\mathrm{d}F_u^i \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p = L_{f_u}(\mathrm{d}z_i) \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p.$$

In view of (6) this becomes

$$\mathrm{d}F_u^i \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p = L_{f_u}(i_X(\omega_{i-p})) \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p.$$

But, $L_{f_u}(i_X(\omega_{i-p})) \in \Omega^X$. So, we can express $L_{f_u}(i_X(\omega_{i-p}))$ as a linear combination of the $n-p$ basis elements of Ω^X where the real coefficients depend on the parameter u. Thus,

$$\mathrm{d}F_u^i \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p = \sum_{j=1}^{n-p} \alpha_j^i(u)\omega_j \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p.$$

One can easily verify the identity $i_X(\omega_j) \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p = \omega_j$, which, again in view of (6), implies that $\omega_j = \mathrm{d}z_{j+p} \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p$, for $j = 1,\ldots,n-p$. Consequently,

$$\mathrm{d}F_u^i \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p = \sum_{j=1}^{n-p} \alpha_j^i(u)\mathrm{d}z_{j+p} \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p.$$

Then, it must be true that

$$\mathrm{d}F_u^i = \sum_{j=1}^{n-p} \alpha_j^i(u)\,\mathrm{d}z_{j+p} + \sum_{j=1}^{p} \phi(z_1,\ldots,z_p,u)\,\mathrm{d}z_j.$$

Note, that it is the integrability requirement that insures that ϕ depends only on the coordinates $z_1,\ldots,z_p$. Integrating, leads to

$$F_u^i = \sum_{j=1}^{n-p} \alpha_j^i(u)z_{j+p} + \varphi(z_1,\ldots,z_p,u), \quad i = p+1,\ldots,n.$$

For $j = 1,\ldots,p$,

$$\mathrm{d}F_u^i \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p = \mathrm{d}L_{f_u}(z_i) \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p.$$

But, in view of Ω_1^X, $\mathrm{d}L_{f_u}(z_i) \in \Omega^X$, for $i = 1,\ldots,p$. So, the remainder of the argument proceeds as before.

Appendix B. Proof of the necessity part of Proposition 3.7

Here, we provide a sketch of the proof. The overall logic follows the arguments of [1] for the single-output case.

The conditions of the theorem are coordinate free. So if the system (1) is transformable to (4) we can verify conditions (1)–(3) in the z-coordinates. We begin by introducing the 'unobservable' distributions (replaces the unobservable subspace of linear systems)

$$\mathscr{F}_0 = \{X | L_X(h_i) = 0,\ i = 1,\ldots,p\},$$

$$\mathscr{F}_{k+1} = \mathscr{F}_k \cap \left\{ X \middle| \begin{array}{c} L_X L_{f_{u^{k+1}}} \cdots L_{f_{u^1}}(h_i) = 0 \\ u^1, \ldots, u^{k+1} \in [0,1]^m \\ i = 1, \ldots, p \end{array} \right\}.$$

Our observability assumption on $\mathscr{E}_0, \ldots . \mathscr{E}_{p^*}$ implies that

$$\mathscr{F}_0 \supset \cdots \supset \mathscr{F}_{p^*} = \{0\}.$$

In z-coordinates, we can compute (following tedious computations as in [1])

$$\mathscr{F}_0 = \{X | CX = 0\},$$
$$\mathscr{F}_{k+1} = \mathscr{F}_k \cap \{X | (CA)_{l_{k+1}} X = 0\}.$$

Accordingly, introduce the sets of constant vector fields

$$F_0 = \{X \in R^n | CX = 0\} = \ker C,$$
$$F_{k+1} = F_k \cap \{X \in R^n | (CA)_{l_{k+1}} X = 0\}.$$

Comparing these, Hammouri and Kinnaert [1] point out that the real vector spaces F_k span the distributions $\mathscr{F}_k$. Their Lemma 5 and Claim 7 are easily extended to the multi-output case:

Lemma 2.13. (i) $\forall u \in \{0,1\}^m$, $[f_u, \mathscr{F}_{k+1}] \subset \mathscr{F}_k$ *for* $k = 0, \ldots, p^* - 1$. (ii) $\forall X \in \mathscr{F}_k \setminus \mathscr{F}_{k+1}$, $\exists u \in \{0,1\}^m$ *such that* $[f_u, X] \in \mathscr{F}_{k-1} \setminus \mathscr{F}_k$ *for* $k = 1, \ldots, p^*$.

Proof. (i) Let $X \in \mathscr{F}_{k+1}$. From the definition of the $\mathscr{F}_k$'s we have for every $(u^1, \ldots, u^r) \in (\{0,1\}^m)^r$ and $1 \leqslant r \leqslant k+1$, $L_X L_{f_{u^r}} \cdots L_{f_{u^1}}(h_i) = 0$, $i = 1, \ldots, p$. Thus,

$$L_{[f_u, X]} L_{f_{u^r}} \cdots L_{f_{u^1}}(h_i) = L_{f_u} L_{f_{u^r}} \cdots L_{f_{u^1}}(h_i) - L_X L_{f_{u^r}} \cdots L_{f_{u^1}}(h_i) = 0$$

so that $[f_u, X] \in \mathscr{F}_k$.

(ii) Assume the contrary, i.e., there exists $X \in \mathscr{F}_k \setminus \mathscr{F}_{k+1}$ such that for every $u \in \{0,1\}^m$, $[f_u, X] \in \mathscr{F}_k$, then using the formula in (i) above, we obtain $L_X L_{f_u} L_{f_{uk}} \cdots L_{f_{u1}}(h_i) = 0$, $i = 1, \ldots, p$, for every $u, u^1, \ldots, u^k \in \{0,1\}^m$. But then, $X \in \mathscr{F}_{k+1}$, which contradicts the assumption. □

On this basis [1] establish the following corresponding result for the F_k.

Lemma 2.14. (i) $\forall u \in \{0,1\}^m, [f_u, F_{k+1}] \subset F_k$, *for* $k = 1, \ldots, p^* - 1$. (ii) $\forall X \in F_k \backslash F_{k+1}, \exists u \in \{0,1\}^m$ *such that* $[f_u, X] \in F_{k-1} \backslash F_k$ *for* $k = 1, \ldots, p^* - 1$.

Proof. For every $X \in F_{k+1}$, $[f_u, X] \in \mathscr{F}_k$, $0 \leqslant k \leqslant p^* - 1$ by Lemma 2.13. Now compute

$$[f_u, X] = -\frac{\partial f_u}{\partial z} X = -A(u)X - \frac{\partial \varphi(y,u)}{\partial y} CX = -A(u)X.$$

Thus, $[f_u, X]$ is a constant vector field. Hence, $[f_u, X] \in F_k$. This proves (i). Now, suppose $X \in F_k \backslash F_{k-1}$. Again from Lemma 2.13, we have $[f_u, X] \in \mathscr{F}_{k-1} \setminus \mathscr{F}_k$. But the calculation above shows $[f_u, X]$ is a constant vector field, so $[f_u, X] \in F_{k-1} \backslash F_k$, thus establishing (ii). □

Proof of main result. Condition (1): by rewriting Eq. (7) in z-coordinates and in view of the definition of the sets F_k, we can establish $Y_i \in F_{\kappa_i-2} \backslash F_{\kappa_i-1}$, for each $i = 1, \ldots, p$.

Lemma 2.14 implies there exists $u^1 \in [0,1]^m$ such that $[f_{u^1}, Y_i] \in F_{\kappa_i-2} \setminus F_{\kappa_i-1}$. Successive application of Lemma 2.14 leads to the conclusion $\exists u_1, \ldots, u_{\kappa_i-1} \in [0,1]^m$ such that

$$[f_{u^{\kappa_i-1}}, [\cdots [f_{u^1}, Y_i] \cdots]] \in F_0 \setminus F_1. \tag{9}$$

Now we can show that there exists u_{κ_i} such that

$$[f_{u_{\kappa_i}}, [\cdots [f_{u_1}, Y] \cdots]] \notin F_0. \tag{10}$$

To see this, assume the contrary, i.e.,

$$\forall u^{\kappa_i} \in R^m; L_{[f_{u^{\kappa_i}}, [\cdots [f_{u^1}, Y_i] \cdots]]}(h) \notin F_0$$

so that

$$L_{f_{u^{\kappa_i}}} L_{[f_{u^{\kappa_i-1}}, [\cdots [f_{u^1}, Y_i] \cdots]]}(h) - L_{[f_{u^{\kappa_i-1}}, [\cdots [f_{u^1}, Y_i] \cdots]]} L_{f_{u^{\kappa_i}}}(h) = 0.$$

Now, since

$$[f_{u_{\kappa_i-1}}, [\cdots [f_{u_1}, Y] \cdots]] \in F_0$$

it follows that $L_{[f_{u_{\kappa_i-1}}, [\cdots [f_{u^1}, Y_i] \cdots]]}(h) = 0$ and, hence,

$$[f_{u^{\kappa_i-1}}, [\cdots [f_{u^1}, Y_i] \cdots]] \in F_1$$

which contradicts Eq. (9). Consequently, (10) holds.

By construction

$$Z^i_{u^1 \ldots u^{\kappa_i-1}} := [f_{u_{\kappa_i-1}}, [\cdots [f_{u_1}, Y_i] \cdots]]$$

is a constant vector field in the z-coordinates, i.e.,

$$Z^i_{u^1 \ldots u^{\kappa_i-1}} = \sum_{k=1}^{n} d^i_k \frac{\partial}{\partial z_k}$$

for some constants d^i_k. Now compute,

$$C^j Z^i_{u^1 \ldots u^{\kappa_i-1}} = C^j \sum_{k=1}^{n} d^i_k \frac{\partial}{\partial z_k} = d^i_j \frac{\partial}{\partial z_j}$$

so that

$$L_{Z^i_{u^1\cdots u^{\kappa_i-1}}}(Cz) = CZ^i_{u^1\cdots u^{\kappa_i-1}} = \begin{bmatrix} d^i_1 \\ \vdots \\ d^i_p \end{bmatrix}.$$

It is not difficult to verify that the (constant) vector fields $Z^i_{u^1\cdots u^{\kappa_i-1}}$ are linearly independent. Now, from (10), we have for each $i = 1,\ldots,p$

$$[\,d^i_1 \quad \cdots \quad d^i_p \quad 0 \quad \cdots \quad 0\,]^{\mathrm{T}} \notin \ker C.$$

Since $\operatorname{rank} C = p$, we have $\dim\ker C = n - p$ and there are precisely p independent vectors not contained in $\ker C$. It follows that the p-vectors $[\,d^i_1 \quad \cdots \quad d^i_p\,]$, $i = 1,\ldots,p$ are independent. Consequently,

$$\left[L_{Z^1_{u^1\cdots u^{\kappa_1-1}}}(Cz) \quad \cdots \quad L_{Z^p_{u^1\cdots u^{\kappa_p-1}}}(Cz)\right] = \begin{bmatrix} d^1_1 & \cdots & d^p_1 \\ \vdots & \ddots & \vdots \\ d^1_p & \cdots & d^p_p \end{bmatrix}$$

is invertible. This implies $\mathscr{I} \neq \emptyset$.

Condition (2): since the $Z^i_{u^1\cdots u^{\kappa_i-1}}$ are constant vector fields in the z-coordinates, Condition (2) holds.

Condition (3): in the z-coordinates, compute

$$X_i = \left[L_{Z^1_{u^1\cdots u^{\kappa_1-1}}}(Cz) \quad \cdots \quad L_{Z^p_{u^1\cdots u^{\kappa_p-1}}}(Cz)\right]^{-1} Z^i_{u^1\cdots u^{\kappa_i-1}} = \frac{\partial}{\partial z_i}$$

which satisfies the conditions of Proposition 3.5 for a system in the form observer form (4), see item (1) in Remark 3.6 and [9,10]. □

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